Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

*Upon the Ruled Surfaces Generated by the Plane Movements whose Centrodes are Congruent Conics Tangent at Homologous Points.**

BY DR. E. M. BLAKE, *Brooklyn, N. Y.*

The movements considered in this paper are defined as follows: Upon a plane α' containing a conic C' moves a coincident plane α, containing a conic C congruent to C', in such a manner that C and C' are always tangent at homologous points, i. e. C and C' are the centrodes of the movement. The locus of a point rigidly attached to α is a curve of the fourth order when C and C' are central conics and of the third order when they are parabolas. The locus is in a plane parallel to α' and the same distance from it that the generating point is from α. The locus of a straight line carried by α and making an angle with it, is a quartic scroll when the centrodes are central conics and a cubic scroll when they are parabolas.

It is the object of the present paper to describe the forms of these scrolls, and the character and situation of their nodal lines and pinch-points. The results are to be regarded from two points; first, as furnishing a method of mechanically generating certain cubic and quartic scrolls; and second, as exhibiting the totality of line loci of the movements considered. In the respects mentioned the results are believed to be new.

The properties of the surfaces are deduced from those of their sections parallel to α', i. e. by viewing the surfaces as built up of the loci of the individual points of the generatrix. The point loci have from time to time been studied, and it has only been found necessary in what follows to state briefly the results obtained.

---

* **Read at the meeting of the American Mathematical Society, Boston, Aug. 19–20, 1898. Thread models of the thirteen types of these surfaces were exhibited.**

The paper contains three sections devoted respectively to the movements whose centrodes are ellipses, hyperbolas and parabolas. The scrolls described in the first two are of the fourth order, and have a nodal circle of infinite radius in the infinitely distant plane parallel to α' and a nodal straight line intersecting it. The scrolls of the third section are cubic with a nodal straight line.

§1.—*The Centrodes are Ellipses.*

It is easily shown* that, if an ellipse roll upon a congruent one, their points of contact being homologous† points of the two ellipses, each focus of the one remains at a constant distance, viz. the transverse axis of one of the ellipses, from a focus of the other. Denoting the foci of C', the ellipse in α', by F_1' and F_2', the corresponding foci of C by F_1 and F_2, the transverse axis of each of the ellipses by $2a$, and the distance between their foci by $2c$; the theorem shows that the movement produced by the rolling ellipses can also be produced by joining the pairs of points F_1, F_1' and F_2, F_2' by links of length $2a$.

This latter method of defining the movement makes it a special case of "three-bar motion." The investigations of Roberts and Cayley‡ have shown that for the general three-bar motion the locus of a point in the plane α is a sextic curve. Under two conditions these degenerate into a quartic and a conic. One condition is here satisfied; the opposite links are equal in length. The conic generated is a circle of radius $2a$, in describing which any straight line of α moves parallel to itself.

We proceed to give the equation and necessary properties of the locus of any point P of α: Let the transverse and conjugate axes of C' be taken for the axes of x and y respectively and a perpendicular to α' through its centre O' for axis of z of a system of coordinates to which any fixed point—α' is regarded as fixed—can be referred. Let the transverse and conjugate axes of C whose center is O, be taken respectively for the axes of ξ and η of a system of coordinates to which points of α can be referred. Further, let F_1' and F_1—foci which

* See Burmester, Kinematik, vol. I, Leipzig, 1888, pp. 302-304, and Clifford, Dynamics, Part I, London, 1898, pp. 146-148.

† Homologous in the sense that by turning one of the ellipses over about a common tangent, the two will be brought into coincidence.

‡ Proc. Lond. Math. Soc., vols. II, III, IV, VII.

remain at the distance $2a$—be respectively upon the positive halves of the axes of x and ξ. The locus of P whose coordinates in α are (ξ, η) is

$$(x^2+y^2)^2-2(\xi^2+\eta^2)(x^2+y^2)-4a^2x^2+4(c^2-a^2)y^2-8a^2\xi x \\ -8\eta(c^2-a^2)y+(\xi^2+\eta^2)^2-4a^2\xi^2+4(c^2-a^2)\eta^2=0, \quad (1)$$

a unicursal curve of the fourth order having (when c is not zero) nodes at the circular points at infinity and a real double point at $(-\xi, \eta)$.* The latter is a node, cusp or conjugate point according as P is without, on or within the ellipse C. The nodes fall on α' outside the ellipse C', the cusps upon it, and the conjugate points within.

It is evident from the mode of generation by the rolling ellipses that a point at a considerable distance from O generates a curve consisting of a loop with a smaller one within joined at a node, similar to the limaçon with a node. As the tracing point still farther recedes from O, the loops become more and more nearly equal passing toward the circular form. The limit for a point at infinity is a double circle. As the tracing point moves from a great distance toward O, both loops of the locus decrease in size, the inner more rapidly than the outer, until, when the ellipse C is reached, the former shrinks to a point. The curve is then similar to the cardioid having a cusp projecting inward. The cusp, as we have seen, is upon C'. Its tangent is normal to C' at the point where the cusp falls. The curve, traced by a point within C, is closed, encircling C' once and having a conjugate point within it. The locus of F_1 is the circle of radius $2a$ whose center is F_1' with the isolated point F_2'; similarly for the locus of F_2.

Under the supposition, as above, that the centrodes are not circles, the only curves having orthogonal symmetry are the loci of the points upon the transverse and conjugate axes of C. They are symmetrical with respect to the corresponding axes of C'. Since they have nodes—not cusps—at the circular points at infinity, they are not limaçon although resembling them. The center O describes the curve

$$(x^2+y^2+2c^2-4a^2)(x^2+y^2)-2c^2(x^2-y^2)=0, \quad (2)$$

symmetrical with respect to both axes.

---

* Roberts, Proc. Lond. Math. Soc., III, 1871, p. 220. Cayley, Ibid., IV, p. 110. Dingeldey, "Ueber die Erzeugung von Curven vierter Ordnung durch Bewegungsmechanismen," Leipzig, 1885, p. 11.

The movement with circular centrodes requires brief consideration. By reducing the focal distances of the elliptical centrodes, they pass finally as a limit into circular ones, the equation (1) applying for c equal to zero. The movement can no longer be regarded as a special case of three-bar motion as two of the links reduce to zero.

With elliptical centrodes there is a twofold infinity of non-congruent curves described by the points of α, but with circular ones the distance of a point from the center of the moving centrode alone defines the form of its locus, all points of a circumference concentric with the centrode tracing congruent curves. These curves, as the equation shows, are limaçon,* and hence have cusps at the circular points at infinity.†

The way is now prepared for readily determining the character of the locus of a line carried by α. The generatrix is assumed to be neither parallel nor perpendicular to it. If parallel, the line would envelope a plane curve the study of which is without the field of this paper. If perpendicular, a right cylinder upon the locus of its piercing point in α is the surface generated. Two parallel generatrices having the same orthogonal projection upon α, evidently generate congruent surfaces which can be brought into coincidence by translation along the axis of z. Further, it is to be remarked that the loci of any two intersecting lines having the same projection upon α can be made congruent by multiplying by a constant the ordinates—that are perpendicular to α'—of one of them. Hence, the position in α of the projection of the generatrix of a surface determines its projective properties.

Let the generatrix l_1 pass through A, the foot of the perpendicular from O to l, its orthogonal projection upon α, and let l_1 make the angle with α whose

---

* Williamson's Diff. Calculus, p. 350, ex. 1.

† There is another movement whose point loci are limaçon. Its centrodes are a circle of radius r fixed, and another of radius $2r$ rolling upon and enclosing it. All the real double points of the curves are upon the circumference of the fixed centrode, whereas, for the movement under consideration, they cover the whole of the plane. Another distinction is as follows: If we regard the limaçon as obtained by extending the radii-vectores of the circle $\rho = 2p \cos\theta$ by a constant quantity m, then when the centrodes are equal circles, m is constant and equal to their radii for all the curves and p is equal to twice the distance of the tracing-point from the center of the moving centrode, while for the movement with unequal centrodes, $p = r$ and m is the variable distance from the tracing-point to the center of the moving centrode. This last movement is discussed by Cayley, "On the Kinematics of a Plane," Quarterly Journal, XVI, 1878, pp. 1–8, and by Schell, "Theorie der Bewegung und der Kräfte, I, Leipzig, 1879, p. 227.

cotangent is s. Take a point P of l whose coordinates are (ξ, η) and let its distance from A be d. Denote the length OA by p and the angle it makes with the axis of ξ by θ, then we have

$$\xi = p\cos\theta - d\sin\theta,$$
$$\eta = p\sin\theta + d\cos\theta.$$

Substituting these values of ξ and η in (1) we obtain the equation of the locus of a point of l at the distance d from A, but it is the orthogonal projection of the curve in the plane $z = \dfrac{d}{s}$ described by that point of l_1 whose projection is P. Hence, the equation of the locus of l_1 is found by eliminating ξ, η and d from (1) by means of the above equations and $d = sz$. The result is

$$\begin{aligned}(x^2+y^2)^2 - 2\,(p^2+s^2z^2)(x^2+y^2) - 4a^2x^2 + 4\,(c^2-a^2)\,y^2 \\ - 8a^2x\,(p\cos\theta - sz\sin\theta) - 8\,(c^2-a^2)\,y\,(p\sin\theta + sz\cos\theta) \\ + (p^2+s^2z^2)^2 - 4a^2\,(p\cos\theta - sz\sin\theta)^2 \\ + 4\,(c^2-a^2)(p\sin\theta + d\cos\theta)^2 = 0,\end{aligned} \tag{3}$$

a surface of the fourth order. It has a nodal circle of infinite radius in the infinitely distant plane that is parallel to α', for the point at infinity upon l describes a double circle. Since the whole of the double circle is actually described, the nodal circle of the surface lies upon it throughout its whole extent. The two pinch-points which are always situated upon this conic are imaginary.* Applying the above transformation to the equations $x = -\xi$, $y = \eta$ defining the position of the real double point of (1), we have

$$x = -p\cos\theta + sz\sin\theta,$$
$$y = \quad p\sin\theta + sz\cos\theta,$$

---

* The most important articles upon the classification quartic scrolls are: Cremona, Mem. della R. Istoria di Bologna, Series II, T. VIII. Cayley, Philos. Trans., 1864, p. 559, and 1869, p. 111. Salmon, Geometry of Three Dimensions, Chap. XVI. Holgate, Amer. Journal, vol. XV, 1893, p. 344. The classifications of the first three authors, except for the order followed, are practically identical. Of the twelve species they recognize, Holgate has treated eight, describing under each a number of subforms. The place in the classifications of Cayley and Holgate of each of the surfaces to be described in this and the following section will be noted.

the equations of a straight line which is a nodal line of the surface. It makes the same angle with α as a generator.

The following types of surfaces will be distinguished:

I. The projection of the generatrix intersects the centrode C in two real and distinct points.

II. The projection of the generatrix is tangent to the centrode.

III. The projection of the generatrix intersects the centrode in two imaginary points.

Those points of l that are without C describe curves whose real double points are nodes, hence the corresponding portion of the nodal straight line lies upon the surface. This portion includes the point at infinity which is the point of intersection of the nodal straight line and the nodal circle. Assuming the conditions for type I, the points of l_1 whose projections are the points of intersection of l and C, describe curves having each a real cusp. The cusps mark pinch-points on the surface. Between them the sections of the surface parallel to α', corresponding to the points of l within C, have each a real conjugate point, showing the nodal line to be isolated. These scrolls are included under Cayley's seventh species and Holgate's F_3^4 subform 3. As l is brought nearer to the position of tangency with C, their two real points of intersection approach, which indicates for the surface generated by l_1 that the pinch-points of the nodal straight line approach each other, shortening its isolated segment until it finally vanishes, giving a surface of type II. These are of Cayley's eleventh species and Holgate's F_4^4. The nodal straight line lies entirely upon the surfaces of type III, its two pinch-points being imaginary. They are hence of Cayley's seventh species and Holgate's F_3^4 subform 5.

The general theory of quartic scrolls having a nodal straight line and nodal conic shows that the former is a generator when its pinch-points coincide (type II), and that when they are separated it is not a generator (types I and III).* This is readily verified in the present instance. From the equations $x = -\xi$, $y = \eta$ connecting the coordinates of the tracing point (ξ, η) in α with those of the double point (x, y) of its locus in α', it follows that the projection of the nodal straight line upon α' occupies the same relative position to C' that l does to C. In other words, by revolving α about a common tangent to C and

* In addition to the references cited see Holgate, Bulletin New York Math. Soc., III, 1894, p. 224.

C' through 180°, the centrodes C and C' are brought into coincidence, and also l and the projection of the nodal straight line. The generatrix l_1 has (without loss of generality) been taken so as to pass through A, the foot of the perpendicular from O upon l, hence the nodal straight line passes through A', the foot of the perpendicular from O' upon its projection in α'. Now, in order that a position of l_1 can coincide with the nodal line, A must fall upon A' and l upon the projection of the nodal line. A moment's consideration convinces one that this can only occur under the conditions for type II. Assume it, and that the projections of the nodal line and generatrix coincide, forming the common tangent of the two centrodes. We have now the nodal line and a generator both passing through the pinch-point whose projection is the point of tangency. As previously remarked, they make the same angle with α', and if they do not coincide there are three sheets of the surface passing through the pinch-point, which is impossible.

Collecting the results obtained, we have the following: *When the centrodes of a plane movement are congruent ellipses tangent at homologous points, the locus of a carried straight line (which is neither parallel nor perpendicular to the plane of the centrodes) is a quartic scroll having a nodal straight line and a nodal circle intersecting it. The latter is of infinite radius and in the plane at infinity. It lies entirely upon the surface. If the projection of the generatrix upon the moving plane intersects the moving centrode in two real and distinct points (type I), the nodal straight line consists of an isolated segment and one lying upon the surface. The latter contains the point at infinity which is its intersection with the nodal circle. Two pinch-points bound the segments. If the projection of the generatrix is tangent to the centrode (type II), the nodal right line lies entirely upon the surface and has upon it a real double pinch-point. If the projection of the generatrix does not intersect the centrode (type III), the nodal straight line lies entirely upon the surface, and the two pinch-points upon it are imaginary. The nodal straight line is a generator in surfaces of type II but not in the others.*

It remains to note a few special varieties of these surfaces. The only ones having planes of symmetry are those whose generatrices project upon the axes of C. When l is the transverse axis, $y = 0$ is the plane of symmetry. The surface has two notable sections parallel to α', described by the points that project into the foci of C. These consist of a circle of radius $2a$ and an isolated point. The center of the circle and the isolated point projected orthogonally upon α'

are the foci of C'. A surface has one such section if l passes through one focus of C. In all surfaces having l passing through O, the mid-section between the pinch-points and parallel to α' is the doubly symmetric curve (2). When l is the conjugate axis, $x = 0$ is the plane of symmetry.

When the centrodes are circles, two generatrices making the same angle with α and whose projections pass the same distance from O, give congruent surfaces. If l passes through O, the surface has a plane of symmetry perpendicular to α' and passing through O' and a single circular section parallel to α'.

§2.—*The Centrodes are Hyperbolas.*

In this section the same notation as in the preceding will be used. The equations of the point and line loci hold good, subject to the condition $c > a$. This movement is also a degenerate case of three-bar motion,* but the fixed link is one of the longer sides of the jointed parallelogram instead of one of the shorter.

The locus of the point (ξ, η) of the plane α is a unicursal quartic having nodes at the circular points at infinity and a real double point at $(-\xi, \eta)$. The latter is a node, cusp or conjugate point according as (ξ, η) is without (on the convex side of), on or within the centrode C, and it falls upon α' in a corresponding position relative to C'.

The curves having a real node consist of two closed loops, mutually exterior, united at the node. They are symmetrical with respect to the transverse or conjugate axis of C' for the tracing point upon the corresponding axis of C. The locus of O, equation (2), is symmetrical with respect to both axes and resembles the lemniscate, which it in fact is, for $c = \sqrt{2}a$.†

As the tracing point approaches the centrode from without, one loop of the curve shrinks to a cusp as the centrode is reached. The curve then consists of a single closed loop with a cusp protruding outwards. The tangent to the cusp is normal to C' at the cusp.

A description of the manner in which the centrodes roll enables one to gain

---

* Burmester, Kinematik, I, pp. 302–304.

† Haedenkamp, Archiv für Math. u. Phys., III, 1843, p. 400.

an insight into the nature of the paths of distant points of α. Starting with two vertices of C and C' in contact as C rolls, the point of contact recedes toward infinity and two asymptotes approach coincidence. The other branches of C and C' are not tangent, but as the asymptotes pass the position of coincidence they become tangent. At the same time the original pair cease to touch. The point of contact now approaches from infinity in the opposite direction to that in which it receded. It finally reaches the vertex opposite to that at which it started, thus completing a half cycle of the movement. Each branch of C rolls only upon a corresponding branch of C'. The movement is the exact analogue of that for the elliptical centrodes; in a complete cycle the point of contact travels once around the curve, and to each point of the fixed centrode corresponds a point of the moving one.

The result of the movement is to make α rotate (about a point remaining within a finite area of α') from a position of coincidence of two asymptotes to the next such coincidence through an angle equal to twice the angle (that includes a branch) between the asymptotes of a centrode. The next half cycle rotates the plane in the opposite direction to its original position. The locus of a distant point of α must resemble the arc of a circle twice described, which it becomes in the limit for a tracing point at infinity. As equation (1) cannot define a portion of a circle, it gives the whole circumference. Hence a surface generated by a carried straight line has its nodal circle in part upon it and in part isolated.

The curve, by a point within a branch of C, has a conjugate point within the corresponding branch of C', and consists in addition of a closed curve lying without that branch.

The character of the surfaces generated by a carried line can be now readily inferred from the given properties of the point loci as in the preceding section. It will be necessary only to state here the results, which are as follows: *When the centrodes of a plane movement are congruent hyperbolas tangent at homologous points, the locus of a carried straight line (which is neither parallel nor perpendicular to the plane of the centrodes) is a quartic scroll having a nodal straight line and a nodal circle, the two intersecting. The latter is of infinite radius in the plane at infinity. It consists of two segments bounded by pinch-points, one segment lying upon the surface, the other isolated. There are six types of surfaces distinguished by the position of the projection l of the generatrix with respect to the moving centrode C.*

I. l intersects both branches of C in real finite points. The nodal straight line consists of two segments bounded by pinch-points. One is isolated and contains the point at infinity, the other lies upon the surface. The intersection of the two nodal lines is upon the isolated segments of both.

II. l intersects one branch of C in two real finite points. The nodal straight line has two segments separated by pinch-points. The segment lying upon the surface contains the point at infinity which is its intersection with the nodal circle. The point of intersection falls in that segment of the nodal circle that lies upon the surface.

III. l is tangent to C but is not an asymptote. The nodal straight line lies wholly in the surface, and has upon it a double pinch-point. The intersection of the nodal lines is in that segment of the nodal circle that lies upon the surface.

IV. l intersects C in imaginary points. The nodal straight line lies wholly upon the surface, its two pinch-points being imaginary. The intersection of the nodal lines is in that segment of the circle that lies upon the surface.

V. l is parallel to an asymptote of C. The nodal straight line has two segments with two pinch-points bounding them. One of these pinch-points and one upon the nodal circle coincide with the intersection of the nodal lines.

VI. l is an asymptote of C. The nodal straight line lies wholly upon the surface and has a double pinch-point at infinity. This double pinch-point and a pinch-point of the nodal circle coincide with the intersection of the nodal lines.

The nodal straight line in types III and VI is a generator, but in the other types it is not.

The types III and VI belong to Cayley's eleventh species and Holgate's F_4^4. The remaining types are included in Cayley's seventh species. Types I and II belong to Holgate's F_3^4, subform 1; type IV to F_3^4, subform 2; type V to F_3^4, subform 4.*

If l is the transverse axis of C, the surface is symmetrical with respect to $y = 0$, and it has two circular sections parallel to α' and the mid-section (2). When l passes through a single focus, the surface has one circular section. For l coincident with the conjugate axis of C, the surface is symmetrical with respect to $x = 0$.

---

* It is interesting to note that the movements of this and the preceding section furnish a means of mechanically generating examples of all the "subforms"—as recognized by Holgate, Amer. Jour., vol. XV—of scrolls of the fourth order, leaving a nodal conic and nodal straight line.

§3.—*The Centrodes are Parabolas.*

The centrodes are congruent parabolas so placed that, by rolling, their vertices can be brought into coincidence. Let the centrode in α' as before be denoted by C', its focus, vertex and directrix by F', O' and d' respectively, and let the corresponding elements for the other centrode, C, be F, O and d. For axes of x, y and z, take respectively the tangent at the vertex of C', its axis and a perpendicular to α' through O'; and for coordinate axes carried by α take the tangent at the vertex of C for axes of ξ and its axes for axes of η. The positive halves of the axes of y and η are selected so as to pass through F' and F.

It is well known that the focus of each centrode describes the directrix of the other, or conversely, the directrix of each passes through the focus of the other.* Hence, the movement can also be obtained by taking a point F' and a straight line d' in α' and a congruent configuration F, d in α and then conditioning F to remain upon d' and d to pass through F'.† As thus defined, the movement is, however, more general than the preceding. Like the three-bar equivalents of the movements of sections 1 and 2, it is a degenerate case of one of higher order. For it is possible to bring d parallel to d' and have them remain so while α is being translated in their direction. The locus of any point of α is then a straight line parallel to d and d'. Such a straight line with the curve of the third order traced by the same point when the centrodes are the parabolas C and C' constitute a degenerate curve of the fourth order. If the distance from d' to F' is not equal to that from d to F, a point of α in general describes a non-degenerate curve of the fourth order.‡ These curves have been studied by Roberts.||

His paper also treats the cubics resulting by degeneration when the distance from F' to d' is equal to that from F to d, i. e. when the centrodes are congruent parabolas. We proceed to give such of their properties as will be required in

---

* Burmester, Kinematik, I, p. 334.

† The middle point O of the perpendicular from F to d describes the cissoid of Diocles. The first known mention of this method of describing it is in Newton's Arithmetica universalis. See A. v. Braunmühl, "Studie über Curvenerzeugung," in the "Katalog mathematischer Modelle," by Dyck.

‡ When F is upon d we have the mechanism of Nicomedes for drawing the conchoid. Braunmühl in the "Katalog mathematischer Modelle," by Dyck, p. 57.

|| Proc. Lond. Math. Soc., II, 1869, pp. 125–136.

describing the surfaces generated by a carried line. The equation of the locus of the point (ξ, η) is

$$y^3 + x^2y - \eta y^2 + (a - \eta)x^2 - (\xi^2 + \eta^2)y - 2a\xi x + \eta^3 + \eta\xi^2 + a\xi^2 = 0, \quad (4)$$

where a is the distance from the focus to the vertex of the centrodes. The curve is unicursal, having a double point at (ξ, η), which is a node, cusp or conjugate point according as the tracing point is without, on or within C. In a' nodes fall without, cusps upon, and conjugate points within C'. The line $y = \eta - a$ is an asymptote which the curve crosses at the point $\left(\frac{2a\eta + 2\xi^2 - a^2}{2\xi}, \eta - a\right)$. The abscissa of this point is infinite for $\xi = 0$, i. e. for all the curves whose tracing-points are upon the axis of C. The point of contact of the asymptote is an inflexion, and the curves are symmetrical with respect to $x = 0$. The only exception to the condition $\xi = 0$ for an inflexion upon the asymptote is when in addition $\eta = \frac{a}{2}$. The tracing-point is then the focus of C and its locus degenerates to the focus and directrix of C'. The vertex of C, as has been remarked above, is the cissoid.

The equation of the surface generated by a straight line carried by α is obtained by the method of section 1. As there, the generatrix is assumed to be neither perpendicular nor parallel to α. The former would give right cubical cylinders upon (4) as bases, the latter a plane curve. Let s be the cotangent of the angle the generatrix l_1 makes with α, l its projection, and A the foot of the perpendicular from O to l. Denoting the length of OA by p and its angle with the axis of ξ by θ, we have for the coordinates of the point of l distant d from A

$$\xi = p \cos\theta - d \sin\theta,$$
$$\eta = p \sin\theta + d \cos\theta.$$

Eliminating ξ, η and d from (4) by means of these and $d = sz$, we have for the equation of the locus of l_1

$$\begin{aligned} y^3 + x^2y - (p \sin\theta + sz \cos\theta)\, y^2 + (a - p \sin\theta + sz\cos\theta)\, x^2 \\ - (p^2 + s^2z^2)(y - p\sin\theta - sz\cos\theta) - 2ax(p\cos\theta - sz\sin\theta) \\ + a\,(p\cos\theta - sz\sin\theta)^2 = 0. \end{aligned}$$

It is of the third order,* and has a nodal straight line whose equations are

$$x = p \cos\theta - sz \sin\theta,$$
$$y = p \sin\theta + sz \cos\theta.$$

obtained by applying the above transformation to the equations $x = \xi$, $y = \eta$, which define the double point of (4).

After what has been given upon the movement with elliptical centrodes, the following summary of results may be given without further explanation: *When the centrodes of a plane movement are congruent parabolas tangent at homologous points, the locus of a carried line (which is neither perpendicular nor parallel to the plane of the centrodes) is a cubic scroll having a nodal straight line. They are of four types according to the position of the projection l of the generatrix with respect to the moving centrode C.*

I. l intersects C in two real finite points. The nodal line consists of two segments bounded by pinch-points. One segment lies upon the surface and contains the point at infinity, the other is isolated.

II. l is tangent to C. The nodal line lies wholly upon the surface, and has upon it a double pinch-point.

III. l does not intersect C in real points. The nodal line lies wholly upon the surface and its two pinch-points are imaginary.

IV. l is parallel to the axis of C. The nodal line has two segments bounded by pinch-points. One segment lies upon the surface, the other is isolated. Both extend to infinity, since one of the pinch-points is at infinity.

The surfaces for which l is $\xi = 0$ have $x = 0$ a plane of orthogonal symmetry, and are the only ones having such a plane. If l passes through the vertex of C, the section of the surface through the corresponding pinch-point and parallel to α' is the cissoid. If l passes through the focus, one section parallel to α' is a straight line with an isolated point. The first surface mentioned possesses both of these sections.

BROOKLYN, N. Y., *August* 8, 1898.

---

* The order of the surface generated by a carried straight line is not always as low as the apparent order of the point loci of the same movement. Thus the ellipsograph (trammel) gives curves of the second order and surfaces of the fourth. From the point of view of the latter, the ellipses drawn must be regarded as of the fourth order and include the line at infinity taken twice.